NORTH SYDNEY BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 2

2020 HSC Course Assessment Task 3

1st of July, 2020

General instructions

- Working time – 3 hours. (plus 10 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets + answer sheet used in the correct order within this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer sheet provided (numbered as page 13)

SECTION II

- Commence each new question on a new booklet. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

STUDENT NUMBER: .... **# BOOKLETS USED**: .....

Class (please ✔)

◯ 12M4A – Mr Lin ◯ 12M4B – Dr Jomaa ◯ 12M4C – Ms Ziaziaris

Marker's use only.

QUESTION	1-10	11	12	13	14	15	16	**Total**
MARKS	/10	/15	/15	/15	/15	/15	/15	/100

Section I

10 marks
Attempt Question 1 to 10
Allow approximately 15 minutes for this section

Mark your answers on the answer sheet provided.

Questions **Marks**

1. **1**

Given $z = -1 - i$ and $\omega = 1 + i\sqrt{3}$. The Euler form of $\dfrac{z}{\omega}$ is

(A) $\sqrt{2}\,\mathrm{e}^{\frac{11i\pi}{12}}$

(B) $-\sqrt{2}\,\mathrm{e}^{\frac{-i\pi}{12}}$

(C) $-\dfrac{1}{\sqrt{2}}\,\mathrm{e}^{\frac{-i\pi}{12}}$

(D) $\dfrac{1}{\sqrt{2}}\,\mathrm{e}^{\frac{11i\pi}{12}}$

2. **1**

The unit vector in the same direction as $\vec{u} = 4\vec{i} + 4\vec{j} - 2\vec{k}$ is

(A) $\dfrac{-1}{36}\begin{bmatrix} -4 \\ -4 \\ 2 \end{bmatrix}$

(B) $\dfrac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$

(C) $\dfrac{1}{5}\begin{bmatrix} 4 \\ 4 \\ -2 \end{bmatrix}$

(D) $\dfrac{1}{6}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$

3. Which of the following is an expression for $\displaystyle\int \frac{\cos^3 x - \sin^3 x}{\cos x - \sin x}\,dx$? **1**

(A) $x + \dfrac{1}{2}\cos 2x + c$

(B) $x - \dfrac{1}{2}\cos 2x + c$

(C) $x + \dfrac{1}{2}\sin^2 x + c$

(D) $x - \dfrac{1}{2}\sin^2 x + c$

4. A particle of mass m falls vertically from rest under gravity in a medium in which the resistance to motion has magnitude $\dfrac{1}{40}mv^2$ where $v\,ms^{-1}$ is the speed of the particle and $g = 9.8\,ms^{-2}$ is the acceleration due to gravity. What is the terminal velocity of the particle? **1**

(A) $19.8\,ms^{-1}$

(B) $28\,ms^{-1}$

(C) $9.8\,ms^{-1}$

(D) $40\,ms^{-1}$

5. If ω is a complex cube root of unity and $\omega \neq 1$. What is the value of **1**

$$\left(1+\frac{1}{\omega}\right)^{2020}\left(1+\frac{1}{\omega^2}\right)^{2021}.$$

(A) $-\dfrac{1}{\omega}$ (B) ω (C) 0 (D) 1

6. The projection of $\overrightarrow{OA}$ onto $\overrightarrow{OB}$ for $A(4,2,-3)$ and $B(-1,1,1)$. **1**

(A) $\dfrac{5}{3}\begin{bmatrix}1\\-1\\-1\end{bmatrix}$ (C) $\dfrac{5}{3}\begin{bmatrix}4\\2\\-3\end{bmatrix}$

(B) $\dfrac{5}{3}\begin{bmatrix}-1\\1\\1\end{bmatrix}$ (D) $-\dfrac{5}{3}\begin{bmatrix}4\\2\\-3\end{bmatrix}$

7. If the complex number z satisfies $|z| - z - 4(1-2i) = 0$, which of the following is $|z|^2$? **1**

(A) 80 (B) 180 (C) 100 (D) 400

8. A particle of mass m is moving horizontally in a straight line. It experiences a resistive force of magnitude $3m(v+v^2)\,N$ when its speed is v metres per second. At time t seconds, the particle has a displacement of x metres from a fixed point O. Which of the following is the correct expression for x in terms of v? **1**

(A) $x = -\dfrac{1}{3}\displaystyle\int \frac{1}{1+v}\,dv$ (C) $x = -\dfrac{1}{3}\displaystyle\int \frac{1}{v(1+v)}\,dv$

(B) $x = \dfrac{1}{3}\displaystyle\int \frac{1}{1+v}\,dv$ (D) $x = \dfrac{1}{3}\displaystyle\int \frac{1}{v(1+v)}\,dv$

9. The value of $\dfrac{d}{dx}\left(\displaystyle\int_x^{x^2} \frac{1}{t-1}\,dt\right)$ is **1**

(A) $\dfrac{-x}{x^2-1}$ (B) $\dfrac{1}{x-1}$ (C) $\dfrac{1}{x+1}$ (D) $\dfrac{x}{x+1}$

10. **1**

The negation of the following statement:
" $\forall p \in P\,(p\,\text{is of the form}\,4m+1 \Rightarrow p\,\text{can be written as a sum of two squares})$" is

(A) $\forall p \in P, p$ is of the form $4m+1$ and p can not be written as a sum of two squares.

(B) $\exists p \in P, p$ is not of the form $4m+1$ and p can be written as a sum of two squares.

(C) $\forall p \in P, p$ is not of the form $4m+1$ or p can not be written as a sum of two squares.

(D) $\exists p \in P, p$ is of the form $4m+1$ and p can not be written as a sum of two squares.

End of Section I

Section II

90 marks
Attempt Questions 11 to 16
Allow approximately 2 hours and 45 minutes for this section.

Write your answers in the writing booklets supplied. Additional writing booklets are available.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 Marks) Commence a NEW page. **Marks**

(a) Given $z = 2 + 2i\sqrt{3}$ and $\omega = \sqrt{3} + i$

i. Find $z\omega$ in Cartesian form **1**

ii. show that $\dfrac{z}{\omega} = \omega$ **1**

(b) Find $\displaystyle\int \frac{x}{\sqrt{1-x}}\, dx$ **2**

(c) i. Find $\sqrt{6i - 8}$. **2**

ii. Hence, solve the equation **2**

$$2z^2 - (3+i)z + 2 = 0.$$

(d) i. Find a, b and c such that **2**

$$\frac{16}{(x^2+4)(2-x)} = \frac{ax+b}{x^2+4} + \frac{c}{2-x}.$$

ii. Hence, find $\displaystyle\int \frac{1}{(x^2+4)(2-x)}\, dx$ **2**

(e) Given that $\vec{u} = 2\vec{i} + 2\vec{j} - 3\vec{k}$ and $\vec{v} = \vec{i} + 2\vec{j} + 2\vec{k}$

i. Show that $\vec{u}$ and $\vec{v}$ are perpendicular. **1**

ii. Hence, or otherwise find $\vec{w}$ such that **2**

$$|\vec{w}|^2 = |\vec{u}|^2 + |\vec{v}|^2.$$

End of Question 11

Question 12 (15 Marks) Commence a NEW page. **Marks**

(a) Given that m and n are integers.
Statement: "If mn is odd, then m and n are odd".

i. Write down the contrapositive of the statement. **1**

ii. Proving the statement by proving its contrapositive. **2**

(b) The velocity of a particle moving in a straight line is given by

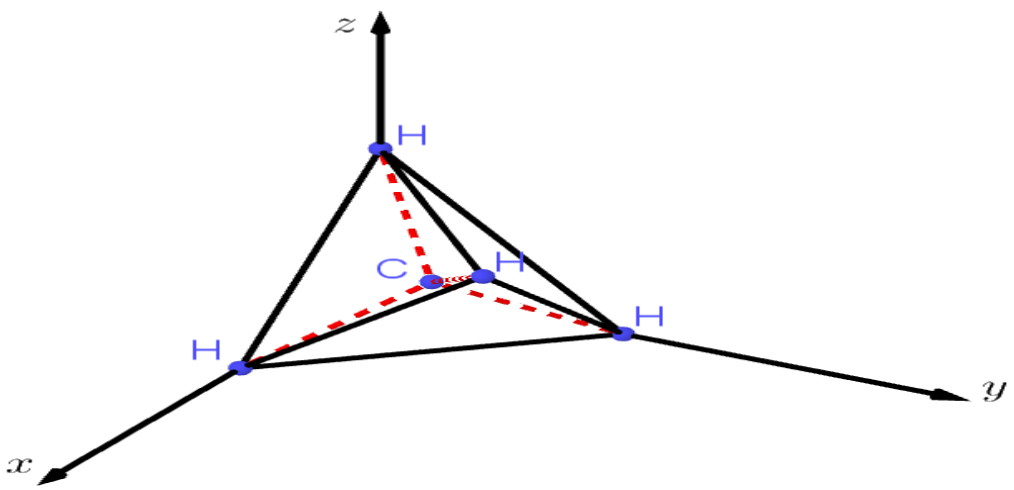
$$v^2 = -9x^2 + 18x + 27.$$

i. Prove that the motion is simple harmonic. **2**

ii. Hence, find the amplitude and the centre of the motion. **2**

iii. Find the maximum acceleration of the particle and state where if occurs. **2**

(c) A molecule of methane CH_4, is structured with the four hydrogen atoms at the vertices of a regular tetrahedron and the carbon atom at the centroid. The bond angle is the angle formed by the $H-C-H$ combination; It is the angle between the lines that join the carbon atom to two of the hydrogen atoms.



Consider the vertices of the tetrahedron to be the points $A(1,0,0), B(0,1,0)$, $D(0,0,1)$ and $E(1,1,1)$ where, A, B, D and E representing the hydrogen atoms as shown in the figure. The carbon atom is represented by point C.

i. Given that **2**

$$\overrightarrow{CA} + \overrightarrow{CB} + \overrightarrow{CD} + \overrightarrow{CE} = 0.$$

Show that the centroid is given by $C(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

ii. Show that the bond angle is about 109.5°. **2**

iii. Find the equation of the line passing through the point C and perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AD}$. **2**

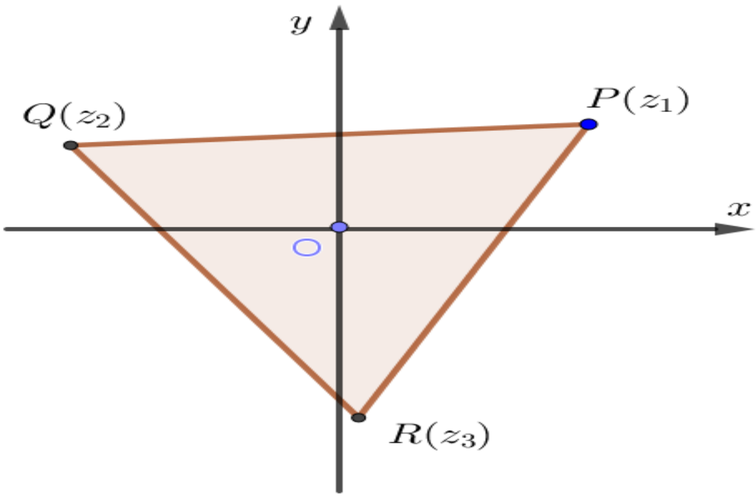
End of Question 12

Question 13 (15 Marks) Commence a NEW page. **Marks**

(a) On an Argand diagram, sketch the region represented by the complex number z where **3**

$$0 \le \arg(z) \le \frac{\pi}{4}, \quad |z-2-2i| \ge 2 \quad \text{and} \quad z+\bar{z} < 8.$$

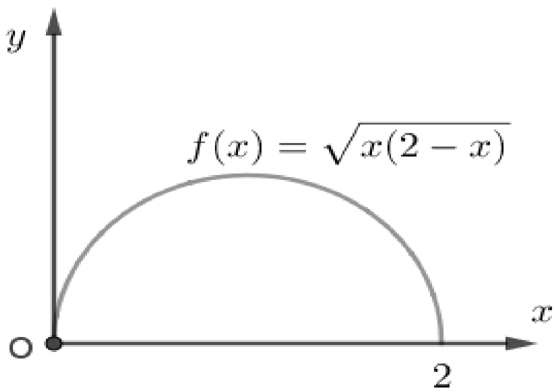
(b) The point P represents the complex number $z_1 = 3+2i$ on the Argand diagram. The complex numbers z_2 and z_3 represented by Q and R respectively, so that PQR is an equilateral triangle whose centre is at the origin. Let $\omega = e^{\frac{i\pi}{3}}$.



i. Show that $z_2 = \omega^2 z_1$ and $z_3 = \omega^4 z_1$ **2**

ii. Find the complex number z_4 represented by the point S such that $PQRS$ is a parallelogram. **2**

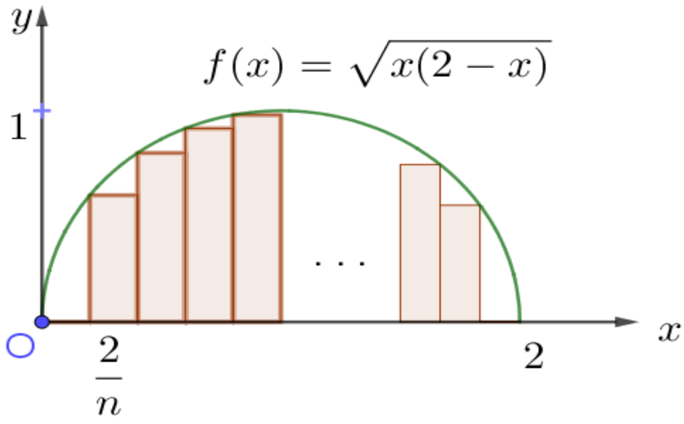
(c) Consider the function $f(x) = \sqrt{x(2-x)}$, see diagram below.



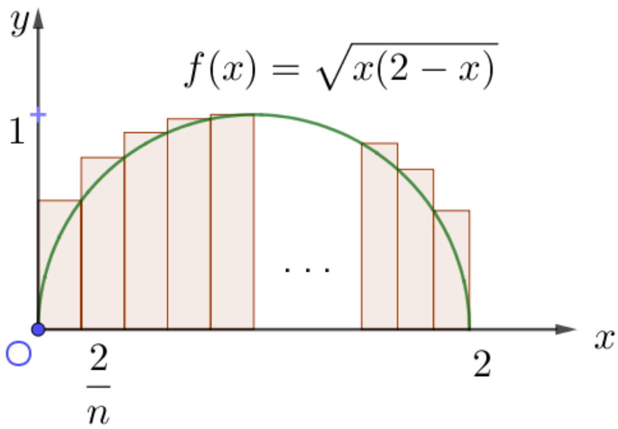
i. Evaluate $\displaystyle\int_0^2 f(x)\,dx$. **1**

Question 13 continues

ii. In the diagram below, the interval $[0, 2]$ is divided into n equal parts. Find the sum of the area of the shaded region. **2**



iii. Find the sum of the area of the shaded region in the diagram below. **2**



iv. Hence or otherwise, show that $\lim_{n\to\infty} \Sigma_{r=1}^{r=n} \dfrac{\sqrt{r(n-r)}}{n^2} = \dfrac{\pi}{8}$ **3**

End of Question 13

Question 14 (15 Marks) Commence a NEW page. **Marks**

(a) A particle A is dropped from a weather balloon. The equation of motion of particle A is

$$\ddot{x} = g - kv_A$$

where g is the acceleration due to gravity, k is a positive constant and v_A is the velocity of particle A.

T seconds later, an identical particle B is projected downwards from the same weather balloon with initial velocity $u\,ms^{-1}$. The equation of motion of particle B is

$$\ddot{x} = g - kv_B$$

where v_B is the velocity of particle B.

i. Show that **4**

$$T = -\frac{1}{k}\ln\left(\frac{g-ku}{g} \times \frac{g-kv_A}{g-kv_B}\right).$$

ii. Show that particle B's displacement x_B is given by **3**

$$x_B = \frac{1}{k}\left[u - v_B + \frac{g}{k}\ln\left(\frac{g-ku}{g-kv_B}\right)\right].$$

iii. Deduce that if particle B catches up with particle A, then particle B must have been released no more than $\dfrac{u}{g}$ seconds after particle A. **3**

(b) A sequence is defined by $y_0 = 2, y_1 = 2\cos x - i\sin x$,

$$y_{n+1} = 2\cos x y_n - y_{n-1}, \quad n \geq 1 \quad \text{and} \quad 0 \leq x \leq \frac{\pi}{2}$$

i. Use mathematical induction to prove that **3**

$$y_n = \frac{1}{2}(\cos x + i\sin x)^n + \frac{3}{2}(\cos x - i\sin x)^n \quad n \geq 0.$$

ii. Hence or otherwise, show that $\lim\limits_{n\to\infty} \dfrac{y_{n+1}}{y_n} = e^{ix}$. **2**

End of Question 14

Question 15 (15 Marks) Commence a NEW page. **Marks**

(a) Consider ω is an $n-$th root of unity. You may assume that

$$1+\omega+\omega^2+\cdots+\omega^{n-1}=0.$$

i. Hence, or otherwise, show that **2**

$$1+2\omega+3\omega^2+\cdots+n\omega^{n-1}=\frac{n}{\omega-1}.$$

ii. By expressing z^n-1 as product of its factors, deduce that **2**

$$(1-\omega)(1-\omega^2)\ldots(1-\omega^{n-1})=n.$$

iii. Let $P(z)=(z-\omega)(z-\omega^2)(z-\omega^3)\ldots(z-\omega^{n-1})$.

A. Find $\dfrac{P'(z)}{P(z)}$. **2**

B. Hence, find **2**

$$\frac{1}{1-\omega}+\frac{1}{1-\omega^2}+\cdots+\frac{1}{1-\omega^{n-1}}.$$

(b) Let $x=\dfrac{a}{a-b}$, $y=\dfrac{b}{b-c}$ and $z=\dfrac{c}{c-a}$, where a,b and c are real numbers.

i. Show that $(x-1)(y-1)(z-1)=xyz$ **2**

ii. Hence or otherwise, show that **2**

$$x+y+z=xy+yz+zx+1.$$

iii. Use (i) and (ii) to show that **3**

$$\left(\frac{2a-b}{a-b}\right)^2+\left(\frac{2b-c}{b-c}\right)^2+\left(\frac{2c-a}{c-a}\right)^2\geq 5.$$

End of Question 15

Question 16 (15 Marks) Commence a NEW page. **Marks**

Given that

$$I_n = \int_0^{\frac{\pi}{4}} (1 + \tan x)^n \, dx$$

i. Show that **4**

$$I_n = \frac{2^{n-1} - 1}{n - 1} + 2I_{n-1} - 2I_{n-2}, \quad n \geq 2$$

ii. Hence or otherwise, evaluate I_5 **3**

iii. Show that **3**

$$I_n = \frac{2^{n-1} - 1}{n - 1} + 2\frac{2^{n-2} - 1}{n - 2} + 2\frac{2^{n-3} - 1}{n - 3} - 4I_{n-4}, \quad n \geq 4$$

iv. Use (iii) to find I_5. **2**

v. Use the substitution $u = \frac{\pi}{4} - x$ to show that **2**

$$I_n = 2^n \int_0^{\frac{\pi}{4}} \left(\frac{1}{1 + \tan x} \right)^n dx$$

vi. Hence or otherwise evaluate **1**

$$\int_0^{\frac{\pi}{4}} \left(\frac{1}{1 + \tan x} \right)^5 dx$$

End of paper.

1. $z = -1 - i = \sqrt{2}\left(\frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = \sqrt{2}\, e^{\frac{5\pi}{4}i}$

$w = 1 + i\sqrt{3} = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 2\, e^{i\pi/3}$

$\frac{z}{w} = \frac{\sqrt{2}\, e^{\frac{5\pi i}{4}}}{2\, e^{i\pi/3}} = \frac{1}{\sqrt{2}} e^{i\left(\frac{5\pi}{4} - \frac{\pi}{3}\right)} = \frac{1}{\sqrt{2}} e^{i\frac{11\pi}{12}}$ (D)

2. $\vec{u} = 4\vec{i} + 4\vec{j} - 2\vec{k}$

$|\vec{u}|^2 = 16 + 16 + 4 = 36 \quad \therefore \quad |\vec{u}| = 6$

$\hat{u} = \frac{1}{6}\begin{bmatrix} 4 \\ 4 \\ -2 \end{bmatrix} \qquad \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ (B)

3. $\int \frac{\cos^3 x - \sin^3 x}{\cos x - \sin x}\, dx = \int (\cos^2 x + \cos x \sin x + \sin^2 x)\, dx$

$= \int (1 + \cos x \sin x)\, dx = x + \frac{\sin^2 x}{2} + C$ (C)

4. $V_T = \sqrt{\frac{g}{k}} = \sqrt{\frac{9.8}{\frac{1}{40}}} = \sqrt{4 \times 98} = 19.8 \text{ m s}^{-1}$ (A)

5. $\left(1 + \frac{1}{w}\right)^{2020}\left(1 + \frac{1}{w^2}\right)^{2021}$

$= \left[\left(1 + \frac{1}{w}\right)\left(1 + \frac{1}{w^2}\right)\right]^{2020}\left(1 + \frac{1}{w^2}\right)$

$1 + w + w^2 = 0$
$1 + w = -w^2$
$1 + w^2 = -w$

$= \left[\frac{1+w}{w} \times \frac{1+w^2}{w^2}\right]^{2020}\left(\frac{1+w^2}{w^2}\right)$

$= \left[\frac{-w^2}{w} \times \frac{-w}{w^2}\right]^{2020}\left(\frac{-w}{w^2}\right)$

$= 1 \times \left(\frac{-1}{w}\right) = \frac{-1}{w}$ (A)

6. $\text{proj}_{\vec{OB}}\,\vec{OA} = \dfrac{(\vec{OA}\cdot\vec{OB})(\vec{OB})}{|\vec{OB}|^2}$

$\vec{OA}\cdot\vec{OB} = 4\times(-1) + 2\times 1 + (-3)\times 1$
$= -4 + 2 - 3$
$= -5$

$|\vec{OB}|^2 = (-1)^2 + 1^2 + 1^2 = 3$

$\text{proj}_{\vec{OB}}\,\vec{OA} = \dfrac{-5}{3}\vec{OB} = \dfrac{-5}{3}\begin{pmatrix}-1\\1\\1\end{pmatrix} = \dfrac{5}{3}\begin{pmatrix}1\\-1\\-1\end{pmatrix}$ A

7. $|z| - z - 4(1-2i) = 0$
$z = |z| - 4 + 8i$
$|z|^2 = (|z|-4)^2 + 64$
$|z|^2 = |z|^2 - 8|z| + 16 + 64$
$\therefore 8|z| = 80 \therefore |z| = 10 \therefore |z|^2 = 100$ C

8. $\dfrac{v\,dv}{dx} = -3v(1+v)$
$-\dfrac{1}{3}\dfrac{dv}{1+v} = dx \therefore x = -\dfrac{1}{3}\displaystyle\int \frac{dv}{1+v}$ A

9. $\dfrac{d}{dx}\displaystyle\int_x^{x^2} \frac{1}{t-1}\,dt = \frac{1}{x^2-1}\times 2x - \frac{1}{x-1}\times 1$
$= \dfrac{2x^2 - 2x - x^2 + 1}{(x^2-1)(x-1)} = \dfrac{x^2-2x+1}{(x^2-1)(x-1)} = \dfrac{(x-1)^2}{(x-1)^2(x+1)} = \dfrac{1}{x+1}$

C

10.

D .

11.

(a) $z = 2 + 2i\sqrt{3} = 4e^{i\pi/3}$

$\omega = \sqrt{3} + i = 2e^{i\pi/6}$

(i) $z\omega = 8e^{i\pi/2} = 8i$

(ii) $\dfrac{z}{\omega} = \dfrac{4e^{i\pi/3}}{2e^{i\pi/6}} = 2e^{i\pi/6} = \omega$

(b)
$$\int \frac{x}{\sqrt{1-x}}\,dx = \int \frac{x-1}{\sqrt{1-x}}\,dx + \int \frac{1}{\sqrt{1-x}}\,dx$$
$$= -\int \sqrt{1-x}\,dx + \int (1-x)^{-1/2}\,dx$$
$$= \frac{2}{3}(1-x)^{3/2} - 2(1-x)^{1/2} + c$$
$$= 2\sqrt{1-x}\left(\frac{1-x}{3} - 1\right) + c$$
$$= 2\sqrt{1-x}\left(\frac{1-x-3}{3}\right) + c$$
$$= \frac{-2}{3}(x+2)\sqrt{1-x} + c$$

(c)

(i) $\sqrt{6i-8} = \sqrt{1^2 - 3^2 + 2 \times 1 \times 3 \times i} = \sqrt{(1+3i)^2} = \pm(1+3i)$

(ii) $2z^2 - (3+i)z + 2 = 0$

$\Delta = [-(3+i)]^2 - 4 \times 2 \times 2$

$= 9 + 6i - 1 - 16$

$= 6i - 8$

$z = \dfrac{3+i \pm (1+3i)}{4}$ $\begin{cases} 1+i \\ \dfrac{1-i}{2} \end{cases}$

(d) (i) $$\frac{16}{(x^2+4)(2-x)} = \frac{ax+b}{x^2+4} + \frac{c}{2-x}$$

$$16 = (ax+b)(2-x) + c(x^2+4)$$
$$= 2ax + 2b - ax^2 - bx + cx^2 + 4c$$
$$= (c-a)x^2 + (2a-b)x + 2b + 4c$$

$c - a = 0 \quad \therefore \quad c = a$

$2a - b = 0 \quad \therefore \quad b = 2a$

$2b + 4c = 16 \quad \therefore \quad b + 2c = 8 \quad \therefore \quad 2a + 2a = 8 \quad \therefore \quad a = 2$

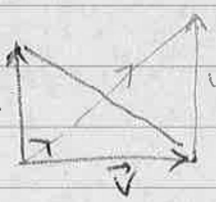
$\therefore \quad c = 2$ and $b = 4$

(ii) $$\int \frac{1}{(x^2+4)(2-x)}\,dx = \frac{1}{16}\int \frac{16}{(x^2+4)(2-x)}\,dx$$
$$= \frac{1}{16}\left[\int \frac{2x+4}{x^2+4}\,dx + 2\int \frac{dx}{2-x}\right]$$
$$= \frac{1}{16}\left[\int \frac{2x}{x^2+4}\,dx + \int \frac{4}{x^2+4}\,dx + 2\int \frac{dx}{2-x}\right]$$
$$= \frac{1}{16}\left[\ln(x^2+4) + 4 \times \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) - 2\ln|2-x|\right] + c$$
$$= \frac{1}{16}\ln(x^2+4) + \frac{1}{8}\tan^{-1}\left(\frac{x}{2}\right) - \frac{1}{8}\ln|2-x| + c$$

(e) $\vec{u} = 2\vec{i} + 2\vec{j} - 3\vec{k} \qquad \vec{v} = \vec{i} + 2\vec{j} + 2\vec{k}$

(i) $$\vec{u}\cdot\vec{v} = 2(1) + 2(2) + (-3)(2)$$
$$= 2 + 4 - 6$$
$$= 0 \qquad \therefore \quad \vec{u} \perp \vec{v}$$

(ii) $\vec{u}$, $\vec{v}$ and $\vec{w}$ are the sides of right angle triangle, where $\vec{u}$ $\vec{w}$ is the hypotenuse



or $\vec{w} = \vec{u} + \vec{v} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$

$\vec{u} + \vec{w} = \vec{v}$ or $\vec{v} + \vec{w} = \vec{u}$

ALSO $-(\vec{u}+\vec{v})$

$\vec{w} = \vec{v} - \vec{u}$ or $\vec{w} = \vec{u} - \vec{v} \qquad \therefore \quad \vec{w} = \pm\begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix}$

12.
(a) If mn is odd then m and n are odd
(i) contrapositive

If m or n are even, then mn is even.

(ii) m is even and n is even then mn is even
(product of even is even).

if m is even and n is odd
$m = 2k$ and $n = 2p+1$

$mn = 2k(2p+1) = 2(2kp+k) = 2q$ is even.

(b) $v^2 = -9x^2 + 18x + 27 = -9(x^2 - 2x - 3) = -9(x-3)(x+1)$

(i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{2}\left(-9(2x) + 18\right)$
$= -9x + 9$
$= -3^2(x-1)$

(ii) centre $x = 1$, amplitude $= 2$

(iii) Maximum acceleration $\therefore v = 0$
$\therefore x = -1$ or $x = 3$

at $x = -1$, $\ddot{x} = 18$

at $x = 3$, $\ddot{x} = -18$.

(c)
(i) $\vec{OA} - \vec{OC} + \vec{OB} - \vec{OC} + \vec{OD} - \vec{OC} + \vec{OE} - \vec{OC} = 0$
$\vec{OA} + \vec{OB} + \vec{OD} + \vec{OE} = 4\vec{OC}$
$\begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\end{pmatrix} + \begin{pmatrix}1\\1\\1\end{pmatrix} = 4\vec{OC}$
$\begin{pmatrix}2\\2\\2\end{pmatrix} = 4\vec{OC}$
$\vec{OC} = \frac{1}{4}\begin{pmatrix}2\\2\\2\end{pmatrix} = \begin{pmatrix}1/2\\1/2\\1/2\end{pmatrix} \quad \therefore \quad C\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$.

(ii) $\vec{CA} = \vec{OA} - \vec{OC} = \begin{pmatrix}1\\0\\0\end{pmatrix} - \begin{pmatrix}1/2\\1/2\\1/2\end{pmatrix} = \begin{pmatrix}1/2\\-1/2\\-1/2\end{pmatrix}$

$\vec{CB} = \vec{OB} - \vec{OC} = \begin{pmatrix}0\\1\\0\end{pmatrix} - \begin{pmatrix}1/2\\1/2\\1/2\end{pmatrix} = \begin{pmatrix}-1/2\\1/2\\-1/2\end{pmatrix}$

$|\vec{CA}| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{3/4} = \frac{\sqrt{3}}{2}$

$|\vec{CB}| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{3/4} = \frac{\sqrt{3}}{2}$

$\vec{CA} \cdot \vec{CB} = \frac{1}{2} \times \left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right) \times \left(\frac{1}{2}\right) + \left(-\frac{1}{2}\right) \times \left(-\frac{1}{2}\right) = -1/4$

$\cos\angle ACB = \dfrac{\vec{CA} \cdot \vec{CB}}{|\vec{CA}|\,|\vec{CB}|} = \dfrac{-1/4}{\frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2}} = \dfrac{-1}{3}$

$\angle ACB = 109^\circ 28'$
$= 109.47^\circ$
$= 109.5^\circ$

(iii) $\vec{CE} = \vec{OE} - \vec{OC} = \begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}1/2\\1/2\\1/2\end{pmatrix} = \begin{pmatrix}1/2\\1/2\\1/2\end{pmatrix}$

$\vec{AB} = \vec{OB} - \vec{OA} = \begin{pmatrix}0\\1\\0\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}-1\\1\\0\end{pmatrix}$

$\vec{AD} = \vec{OD} - \vec{OA} = \begin{pmatrix}0\\0\\1\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}-1\\0\\1\end{pmatrix}$

$\vec{AB} \cdot \vec{CE} = -1 \times \frac{1}{2} + 1 \times \frac{1}{2} + 0 \times \frac{1}{2} = 0 \therefore \vec{AB} \perp \vec{CE}$

$\vec{AD} \cdot \vec{CE} = -1 \times \frac{1}{2} + 0 \times \frac{1}{2} + 1 \times \frac{1}{2} = 0 \quad \therefore \vec{AD} \perp \vec{CE}$

$\vec{CE} \perp \vec{AB}$ and $\vec{AD}$ $\therefore \vec{CE} \parallel$ line

Equation of line $= \begin{pmatrix}\frac{1}{2}\\1/2\\1/2\end{pmatrix} + t\begin{pmatrix}1/2\\1/2\\1/2\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\1/2\\1/2\end{pmatrix}(1+t) \quad t \in \mathbb{R}.$

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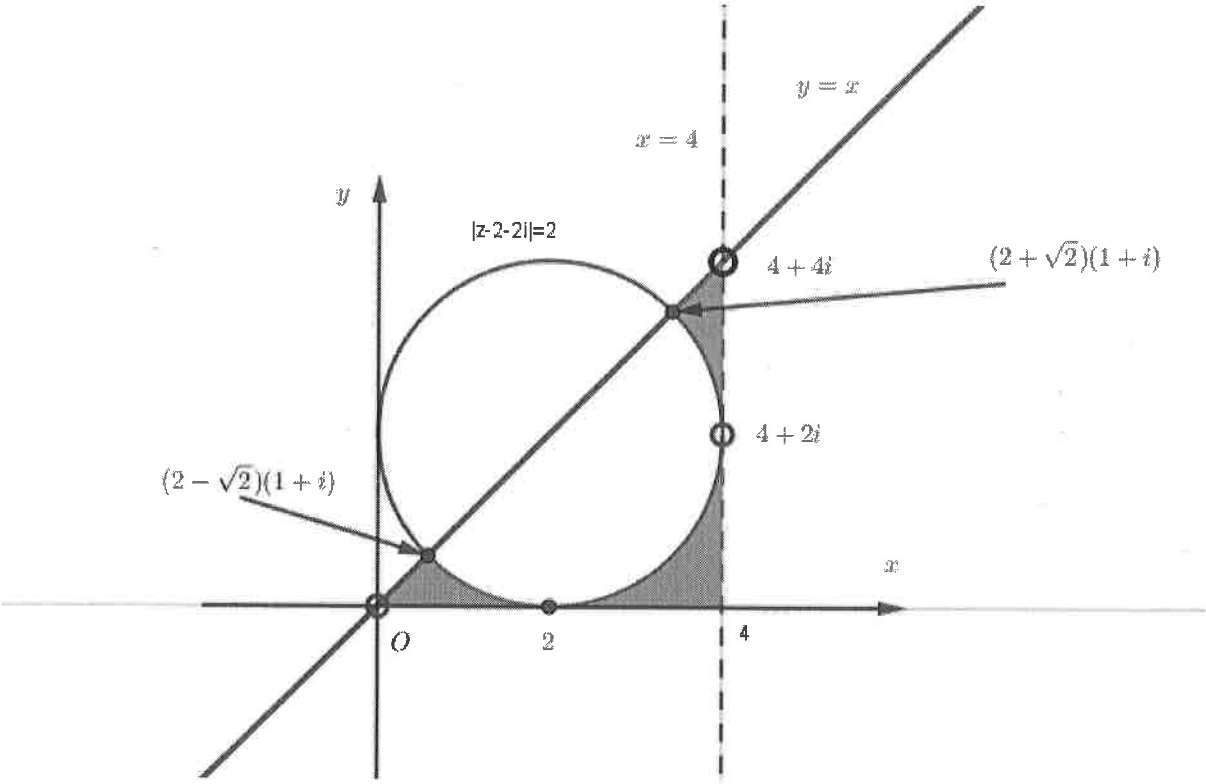
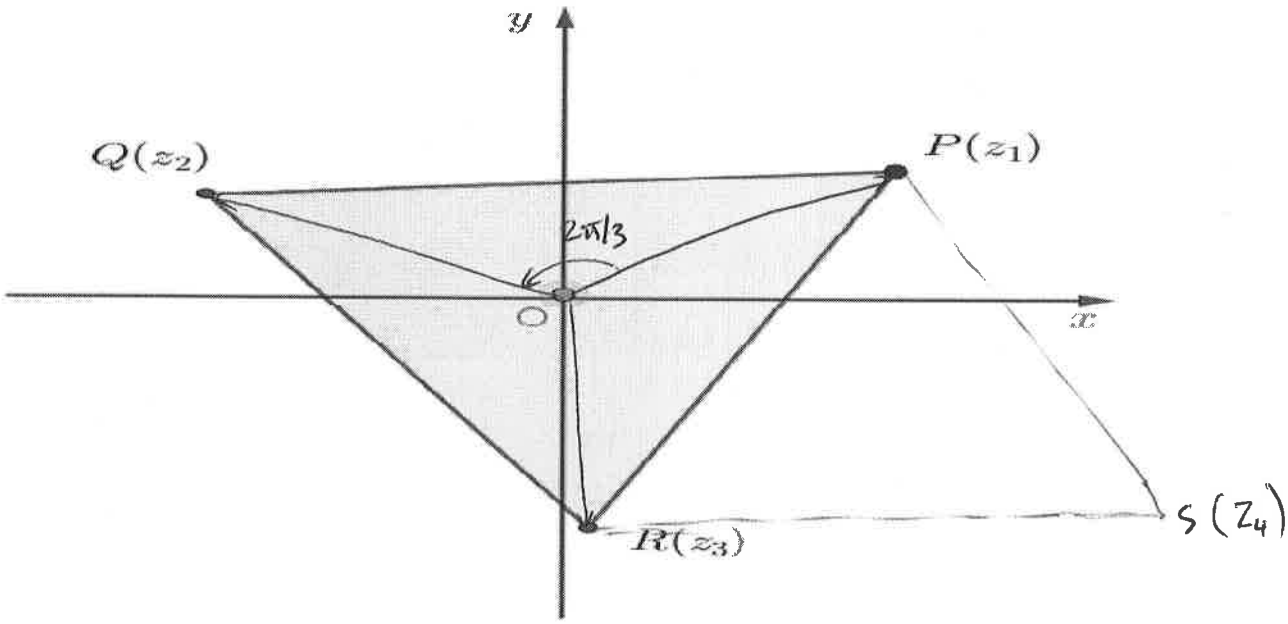
a.



$y = x$
$x = 4$
y
|z-2-2i|=2
$4+4i$
$(2+\sqrt{2})(1+i)$
$4+2i$
$(2-\sqrt{2})(1+i)$
x
O
2
4

b.



y
$Q(z_2)$
$P(z_1)$
$2\pi/3$
O
x
$S(Z_4)$
$R(z_3)$

(i) $\overrightarrow{OQ} = \overrightarrow{OP} \operatorname{cis} 2\pi/3 = \overrightarrow{OP}\left(e^{i\pi/3}\right)^2 = \overrightarrow{OP} \cdot \omega^2$

$$z_2 = z_1 \omega^2$$

Similarly, $\overrightarrow{OR} = \overrightarrow{OQ}\, \omega^2$

$$z_3 = z_1 \omega^2 \omega^2 = z_1 \omega^4$$

(ii) PQRS is a parallelogram $\therefore$

$$\overrightarrow{QP} = \overrightarrow{RS}$$
$$\overrightarrow{OP} - \overrightarrow{OQ} = \overrightarrow{OS} - \overrightarrow{OR}$$
$$\overrightarrow{OS} = \overrightarrow{OP} - \overrightarrow{OQ} + \overrightarrow{OR}$$
$$= z_1 - z_1\omega^2 + z_1\omega^4$$
$$= z_1(1 - \omega^2 + \omega^4)$$

ω cube root of unity $\therefore \omega^3 = 1$ and $\omega^4 = \omega$

Also $1 + \omega + \omega^2 = 0 \therefore 1 + \omega = -\omega^2$

$$1 - \omega^2 + \omega^4 = -2\omega^2$$

$$z_4 = z_1(-2\omega^2) = -2\omega^2 z_1$$

$$z_4 = (3 + 2\sqrt{3}) + (2 - 3\sqrt{3})i$$

(c)

(i) $\displaystyle\int_0^2 f(x)\,dx = \frac{1}{2}\pi(1)^2 = \pi/2$

(area of semi circle with radius 1) (1)

(ii) $\displaystyle\frac{2}{n}\left[f\left(\frac{2}{n}\right)+f\left(\frac{4}{n}\right)+\cdots+f\left(\frac{2n-2}{n}\right)\right]-\frac{2}{n}f(1)$

$\displaystyle= \frac{2}{n}\sum_{k=1}^{n-1}\sqrt{\frac{2k}{n}\left(2-\frac{2k}{n}\right)}-\frac{2}{n}f(1)$

$\displaystyle= \frac{2}{n}\sum_{k=1}^{n-1}\frac{2}{n}\sqrt{k(n-k)}-\frac{2}{n}$

$\displaystyle= 4\sum_{k=1}^{n-1}\frac{\sqrt{k(n-k)}}{n^2}-\frac{2}{n}$ (2)

(iii) $\displaystyle\frac{2}{n}\left[f\left(\frac{2}{n}\right)+f\left(\frac{4}{n}\right)+\cdots+f\left(\frac{2n-2}{n}\right)\right]+\frac{2}{n}f(1)$

$\displaystyle= \frac{2}{n}\sum_{k=1}^{n-1}\sqrt{\frac{2k}{n}\left(2-\frac{2k}{n}\right)}+\frac{2}{n}f(1)$

$\displaystyle= 4\sum_{k=1}^{n-1}\frac{\sqrt{k(n-k)}}{n^2}+\frac{2}{n}$ (2)

(iv) Area from (ii) $\leq$ Area of semi circle $\leq$ Area from iii

$$4\sum_{k=1}^{n-1}\frac{\sqrt{k(n-k)}}{n^2} - \frac{2}{n} \leq \frac{1}{2}\pi \leq 4\sum_{k=1}^{n-1}\frac{\sqrt{k(n-k)}}{n^2} + \frac{2}{n}$$

replace k by r (Note: As $n \to \infty$, $\frac{2}{n} \to 0$)

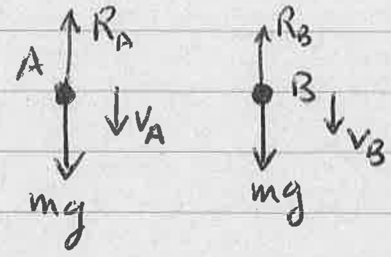
$$\therefore \lim_{n\to\infty} 4\sum_{r=1}^{n}\frac{\sqrt{r(n-r)}}{n^2} \leq \frac{1}{2}\pi \leq \lim_{r\to\infty} 4\sum_{r=1}^{n}\frac{\sqrt{r(n-r)}}{n^2}$$

$$\therefore 4\lim_{n\to\infty}\sum_{r=1}^{n}\frac{\sqrt{r(n-r)}}{n^2} = \frac{1}{2}\pi$$

$$\therefore \lim_{n\to\infty}\sum_{r=1}^{n}\frac{\sqrt{r(n-r)}}{n^2} = \frac{1}{8}\pi.$$

3

Question 14:



(a)

$$\ddot{x} = g - k v_A$$

$t = 0$, $\dot{x} = 0$
$x = 0$

$$\dot{v}_A = g - k v_A$$

$$\frac{k \dot{v}_A}{g - k v_A} = k$$

$$\int \frac{-k \dot{v}_A \, dt}{g - k v_A} = \int -k \, dt$$

$$\log(g - k v_A) = -kt + c_1$$

At $t = 0 \therefore c_1 = \log g$

$$\log(g - k v_A) = -kt + \log g$$

$$\log\left(\frac{g - k v_A}{g}\right) = -kt \quad \therefore \quad t = \frac{-1}{k} \log\left(\frac{g - k v_A}{g}\right) \quad (1)$$

$$\ddot{x} = g - k v_B$$

$t = 0$, $\dot{x} = u$
$x = 0$

$$\frac{k \dot{v}_B}{g - k v_B} = k \quad \therefore \int \frac{-k \dot{v}_B \, dt}{g - k v_B} = \int -k \, dt$$

$$\log(g - k v_B) = -kt + c_2$$

At $t = 0$, $c_2 = \log(g - ku)$

$$\log(g - kv_B) = -kt + \log(g - ku)$$

$$kt = \log(g - ku) - \log(g - kv_B)$$

$$= \log\left(\frac{g - ku}{g - kv_B}\right)$$

$$t = \frac{1}{k}\log\left(\frac{g - ku}{g - kv_B}\right) \quad (2)$$

$(1) - (2)$ $\therefore$

$$T = \frac{-1}{k}\log\left(\frac{g - kv_A}{g}\right) - \frac{1}{k}\log\left(\frac{g - ku}{g - kv_B}\right)$$

$$= \frac{-1}{k}\log\left(\frac{g - ku}{g} \times \frac{g - kv_A}{g - kv_B}\right).$$

(ii) $\ddot{x} = g - kv_B$

$$v_B \frac{dv_B}{dx} = g - kv_B$$

$$\frac{v_B \, dv_B}{g - kv_B} = dx$$

$$\frac{-kv_B \, dv_B}{g - kv_B} = -k\,dx$$

$$\frac{(g - kv_B - g)\,dv_B}{g - kv_B} = -k\,dx$$

$$dv_B - \frac{g}{g - kv_B}\,dv_B = -k\,dx$$

$$v_B + \frac{g}{k}\ln(g - kv_B) = -kx_B + C_3$$

$t = 0 \quad v_B = u \;, \quad x = 0$

$$C_3 = u + \frac{g}{k}\ln(g - ku)$$

$$kx = u + \frac{g}{k}\ln(g-ku) - v_B - \frac{g}{k}\ln(g - kv_B)$$

$$= u - v_B + \frac{g}{k}\ln\left(\frac{g-ku}{g-kv_B}\right)$$

$$x_B = \frac{1}{k}\left[u - v_B + \frac{g}{k}\ln\left(\frac{g-ku}{g-kv_B}\right)\right].$$

(iii) use the result of (ii) for particle A

$$x_A = \frac{1}{k}\left[-v_A + \frac{g}{k}\ln\left(\frac{g}{g-kv_A}\right)\right]$$

set $x_A = x_B$

$$u - v_B + \frac{g}{k}\ln\left(\frac{g-ku}{g-kv_B}\right) = -v_A + \frac{g}{k}\ln\left(\frac{g}{g-kv_A}\right)$$

$$\frac{g}{k}\ln\left(\frac{g-ku}{g-kv_B}\right) - \frac{g}{k}\ln\left(\frac{g}{g-kv_A}\right) = v_B - v_A - u$$

$$\frac{g}{k}\ln\left(\frac{g-ku}{g}\times\frac{g-kv_A}{g-kv_B}\right) = v_B - v_A - u$$

$$-\frac{1}{k}\ln\left(\frac{g-ku}{g}\times\frac{g-kv_A}{g-kv_B}\right) = +\frac{u}{g} \quad \left(\text{at point of contact}\right)$$

$$T = \frac{u}{g} \quad (\text{using } T \text{ from (i)}).$$

(b) (i)

$$y_0 = \frac{1}{2}(\cos x + i\sin x)^0 + \frac{3}{2}(\cos x - i\sin x)^0$$
$$= \frac{1}{2} + \frac{3}{2} = \frac{4}{2} = 2$$

$$y_1 = \frac{1}{2}(\cos x + i\sin x) + \frac{3}{2}(\cos x - i\sin x)$$
$$= 2\cos x - i\sin x$$

Assume it is true for $n = k$ and $n = k-1$

$$y_k = \frac{1}{2}(\cos x + i\sin x)^k + \frac{3}{2}(\cos x - i\sin x)^k$$
$$= \frac{1}{2}(e^{ix})^k + \frac{3}{2}(e^{-ix})^k$$

$$y_k = \frac{1}{2}e^{ikx} + \frac{3}{2}e^{-ikx} \quad \text{and} \quad y_{k-1} = \frac{1}{2}e^{i(k-1)x} + \frac{3}{2}e^{-i(k-1)x}$$

Prove it true for $n = k+1$

$$y_{k+1} = 2\cos x\, y_k - y_{k-1}$$
$$= 2\cos x\left(\frac{1}{2}e^{ikx} + \frac{3}{2}e^{-ikx}\right) - \frac{1}{2}e^{i(k-1)x} - \frac{3}{2}e^{-i(k-1)x}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \quad \therefore 2\cos x = e^{ix} + e^{-ix}$$

$$y_{k+1} = \left(e^{ix} + e^{-ix}\right)\left(\frac{1}{2}e^{ikx} + \frac{3}{2}e^{-ikx}\right) - \frac{1}{2}e^{i(k-1)x} - \frac{3}{2}e^{-i(k-1)x}$$

$$= \frac{1}{2}e^{i(k+1)x} + \frac{1}{2}e^{i(k-1)x} + \frac{3}{2}e^{-i(k-1)x} + \frac{3}{2}e^{-i(k+1)x} - \frac{1}{2}e^{i(k-1)x} - \frac{3}{2}e^{-i(k-1)x}$$

$$= \frac{1}{2}e^{i(k+1)x} + \frac{3}{2}e^{-i(k+1)x}$$

True for $n = k+1$

Hence by mathematical induction it is true for all $n \geq 0$.

(ii) $$\lim_{n\to\infty} \frac{y_{n+1}}{y_n} = \lim_{n\to\infty} \frac{\frac{1}{2}e^{i(n+1)x} + \frac{3}{2}e^{-i(n+1)x}}{\frac{1}{2}e^{inx} + \frac{3}{2}e^{-inx}}$$

$$= \lim_{n\to\infty} \frac{e^{i(n+1)x}\left[\frac{1}{2} + \frac{3}{2}e^{-2i(n+1)x}\right]}{e^{inx}\left[\frac{1}{2} + \frac{3}{2}e^{-2inx}\right]}$$

$$= e^{ix} \lim_{n\to\infty} \frac{\frac{1}{2} + \frac{3}{2}e^{-2i(n+1)x}}{\frac{1}{2} + \frac{3}{2}e^{-2inx}}$$

$$= e^{ix}.$$

Question 15:

(a) (i) $S = 1 + 2\omega + 3\omega^2 + \cdots + n\omega^{n-1}$

$\omega S = \omega + 2\omega^2 + \cdots + (n-1)\omega^{n-1} + n\omega^n$

$S - \omega S = 1 + \omega + \omega^2 + \cdots + \omega^{n-1} - n\omega^n$

$= -n\omega^n$ since $(1 + \omega + \cdots + \omega^{n-1} = 0)$

$S(1-\omega) = -n\omega^n = -n$ since $\omega^n = 1$

$S = \dfrac{n}{\omega - 1}$

(ii) $z^n - 1 = (z-1)(z-\omega)(z-\omega^2)\cdots(z-\omega^{n-1})$

$\dfrac{z^n - 1}{z-1} = (z-\omega)(z-\omega^2)\cdots(z-\omega^{n-1})$

$1 + z + z^2 + \cdots + z^{n-1} = (z-\omega)(z-\omega^2)\cdots(z-\omega^{n-1})$

For $z = 1$, $n = (1-\omega)(1-\omega^2)\cdots(1-\omega^{n-1})$

(iii) $p(z) = (z-\omega)(z-\omega^2)\cdots(z-\omega^{n-1})$

$\dfrac{p'(z)}{p(z)} = \dfrac{1}{z-\omega} + \dfrac{1}{z-\omega^2} + \cdots + \dfrac{1}{z-\omega^{n-1}}$

$p(z) = 1 + z + z^2 + \cdots + z^{n-2} + z^{n-1}$

$p'(z) = 1 + 2z + 3z^2 + \cdots + (n-2)z^{n-3} + (n-1)z^{n-2}$

$p'(1) = 1 + 2 + 3 + \cdots + n-1 = \dfrac{(n-1)n}{2}$

$p(1) = n$

$\dfrac{p'(1)}{p(1)} = \dfrac{\frac{(n-1)n}{2}}{n} = \dfrac{n-1}{2} \therefore \dfrac{1}{1-\omega} + \dfrac{1}{1-\omega^2} + \cdots + \dfrac{1}{1-\omega^{n-1}} = \dfrac{n-1}{2}.$

(b) $x = \frac{a}{a-b}$, $y = \frac{b}{b-c}$, $z = \frac{c}{c-a}$

(i) $$x-1 = \frac{a}{a-b} - 1 = \frac{a-a+b}{a-b} = \frac{b}{a-b}$$

$$y-1 = \frac{b}{b-c} - 1 = \frac{b-b+c}{b-c} = \frac{c}{b-c}$$

$$z-1 = \frac{c}{c-a} - 1 = \frac{c-c+a}{c-a} = \frac{a}{c-a}$$

$$(x-1)(y-1)(z-1) = \frac{b}{a-b} \times \frac{c}{b-c} \times \frac{a}{c-a}$$

$$= \frac{a}{a-b} \times \frac{b}{b-c} \times \frac{c}{c-a}$$

$$= xyz$$

(ii) $$(x-1)(y-1)(z-1) = xyz$$

$$(x-1)(yz-y-z+1) = xyz$$

$$xyz - xy - xz + x - yz + y + z - 1 = xyz$$

$$x + y + z = xy + yz + zx + 1 .$$

(iii) $$\left(\frac{2a-b}{a-b}\right)^2 + \left(\frac{2b-c}{b-c}\right)^2 + \left(\frac{2c-a}{c-a}\right)^2$$

$$= \left(1 + \frac{a}{a-b}\right)^2 + \left(1 + \frac{b}{b-c}\right)^2 + \left(1 + \frac{c}{c-a}\right)^2$$

$$= (1+x)^2 + (1+y)^2 + (1+z)^2$$

$$= 1 + 2x + x^2 + 1 + 2y + y^2 + 1 + 2z + z^2$$

$$= 3 + x^2 + y^2 + z^2 + 2(x+y+z)$$

$$= 3 + x^2 + y^2 + z^2 + 2(xy + yz + zx + 1)$$

$$= 5 + x^2 + y^2 + z^2 + 2(xy + yz + zx)$$

$$= 5 + (x+y+z)^2 \geqslant 5 .$$

Question 16:

$$I_n = \int_0^{\pi/4} (1+\tan x)^n \, dx$$

(i) $$I_n = \int_0^{\pi/4} (1+\tan x)^{n-2} (1+\tan x)^2 \, dx$$

$$= \int_0^{\pi/4} (1+\tan x)^{n-2} (1+2\tan x+\tan^2 x) \, dx$$

$$= \int_0^{\pi/4} (1+\tan x)^{n-2} (1+\tan^2 x) \, dx + 2\int_0^{\pi/4} \tan x (1+\tan x)^{n-2} dx$$

$$= \left[\frac{(1+\tan x)^{n-1}}{n-1} \right]_0^{\pi/4} + 2\int_0^{\pi/4} (1+\tan x - 1)(1+\tan x)^{n-2} dx$$

$$= \frac{(1+\tan \pi/4)^{n-1}}{n-1} - \frac{(1+\tan 0)^{n-1}}{n-1} + 2\int_0^{\pi/4} (1+\tan x)^{n-1} dx - 2\int_0^{\pi/4} (1+\tan x)^{n-2} dx$$

$$= \frac{2^{n-1}-1}{n-1} + 2 I_{n-1} - 2 I_{n-2} .$$

(ii) $I_0 = \int_0^{\pi/4} dx = \pi/4$

$$I_1 = \int_0^{\pi/4} (1+\tan x)\,dx = \left[x - \ln\cos x\right]_0^{\pi/4}$$

$$= \pi/4 - \ln\cos\pi/4 - 0$$

$$= \pi/4 - \ln\frac{1}{\sqrt{2}}$$

$$= \pi/4 + \frac{\ln 2}{2}$$

$$I_2 = \frac{2-1}{1} + 2I_1 - 2I_0$$

$$= 1 + \pi/2 + \ln 2 - 2(\pi/4)$$

$$= 1 + \ln 2$$

$$I_3 = \frac{2^2-1}{2} + 2I_2 - 2I_1$$

$$= \frac{3}{2} + 2 + 2\ln 2 - 2\left(\pi/4 + \frac{\ln 2}{2}\right)$$

$$= \frac{7}{2} + \ln 2 - \pi/2$$

$$I_4 = \frac{2^3-1}{3} + 2I_3 - 2I_2$$

$$= \frac{7}{3} + 7 + 2\ln 2 - \pi - 2 - 2\ln 2$$

$$= \frac{22}{3} - \pi$$

$$I_5 = \frac{2^4-1}{4} + 2I_4 - 2I_3$$

$$= \frac{15}{4} + \frac{44}{3} - 2\pi - 7 - 2\ln 2 + \pi$$

$$\boxed{I_5 = \frac{137}{12} - 2\ln 2 - \pi}$$

(iii) $$I_n = \frac{2^{n-1}-1}{n-1} + 2I_{n-1} - 2I_{n-2} \quad (1)$$

$$I_{n-1} = \frac{2^{n-2}-1}{n-2} + 2I_{n-2} - 2I_{n-3} \quad (2)$$

$$I_{n-2} = \frac{2^{n-3}-1}{n-3} + 2I_{n-3} - 2I_{n-4} \quad (3)$$

(2) + (3) $\therefore$

$$I_{n-1} + I_{n-2} = \frac{2^{n-2}-1}{n-2} + \frac{2^{n-3}-1}{n-3} + 2I_{n-2} - 2I_{n-4}$$

$$\therefore \quad I_{n-1} - I_{n-2} = \frac{2^{n-2}-1}{n-2} + \frac{2^{n-3}-1}{n-3} - 2I_{n-4}$$

(1) $\therefore$ $$I_n = \frac{2^{n-1}-1}{n-1} + 2\left(I_{n-1} - I_{n-2}\right)$$

$$= \frac{2^{n-1}-1}{n-1} + 2\left(\frac{2^{n-2}-1}{n-2} + \frac{2^{n-3}-1}{n-3} - 2I_{n-4}\right)$$

$$\boxed{I_n = \frac{2^{n-1}-1}{n-1} + 2 \cdot \frac{2^{n-2}-1}{n-2} + 2 \cdot \frac{2^{n-3}-1}{n-3} - 4I_{n-4}.}$$

(iv) $$I_5 = \frac{2^4-1}{4} + 2 \cdot \frac{2^3-1}{3} + 2 \cdot \frac{2^2-1}{2} - 4I_1$$

$$= \frac{15}{4} + \frac{14}{3} + 3 - 4\left(\pi/4 + \frac{1}{2}\ln 2\right)$$

$$= \frac{137}{12} - 2\ln 2 - \pi.$$

(v) $$I_n = \int_0^{\pi/4} (1+\tan x)^n \, dx$$

$$= \int_{\pi/4}^{0} \left(1+\tan\left(\frac{\pi}{4}-u\right)\right)^n (-du)$$

$$= \int_0^{\pi/4} \left(1+\tan(\pi/4-u)\right)^n du$$

$$= \int_0^{\pi/4} \left(1+\frac{\tan\pi/4-\tan u}{1+\tan\pi/4\tan u}\right)^n du$$

$$= \int_0^{\pi/4} \left(\frac{2}{1+\tan u}\right)^n du$$

$$= 2^n \int_0^{\pi/4} \left(\frac{1}{1+\tan u}\right)^n du$$

$$= 2^n \int_0^{\pi/4} \left(\frac{1}{1+\tan x}\right)^n dx$$

(vi) $$\int_0^{\pi/4} \left(\frac{1}{1+\tan x}\right)^5 dx = \frac{I_5}{2^5}$$

$$= \frac{1}{32}\left(\frac{137}{12} - 2\ln 2 - \pi\right).$$